International Journal of Theoretical Physics, Vol. 39, No. 12, 2000

Probing Quantum Structure with Boolean Localization Systems

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Received February 28, 2000

In an attempt to probe the objects belonging to the quantum species of structure, we develop the idea of using observables of the Boolean species of structures, as coordinatizing objects in the quantum world. This results in a contextualistic perspective on the latter through local Boolean measurement reference frames. The semantics of this representation is discussed extensively.

1. INTRODUCTION

In a previous work [1, 2] we proposed a mathematical scheme for the analysis of quantum event structures based on category-theoretic methods [3–6], and we also attempted an interpretation of the scheme in order to obtain the physical meaning of this construction in relation to the concept of events in quantum theory. The main guiding idea in our investigation has been the use of objects belonging to the Boolean species of event structure as modeling figures for probing the objects belonging to the quantum species of event structure. The language of category theory is perfectly suited to implement this idea in a universal way. The Boolean event algebras shaping objects give rise to structure-preserving maps with these objects as their domains, which under appropriate compatibility relations provide an isomorphism between quantum algebras of events and Boolean localization systems. The essence of this scheme is the development of a Boolean manifold perspective on quantum event structures, according to which a quantum event algebra consists of an interconnected family of Boolean ones interlocking in a nontrivial way.

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0020-7748/00/1200-2761$18.00/0

The physical interpretation of the Boolean manifold scheme takes place through the identification of Boolean charts in systems of localization for quantum event algebras with reference frames, relative to which the results of measurements can be coordinatized. Thus any Boolean chart in an atlas for a quantum algebra of events corresponds to a set of classical Boolean events which become realizable in its experimental context. The above identification is equivalent to the introduction of a relativity principle in quantum theory and suggests a contextualistic interpretation of its formalism. Thus the quantum world is being perceived through Boolean reference frames objectified by measuring arrangements set up experimentally. In this work we attempt to substantiate this interpretational perspective by focusing our attention to the category of quantum observables.

The notion of observable corresponds to a measurable physical quantity in the context of an arrangement set up experimentally. In any experiment performed by an observer the propositions that can be made concerning a physical quantity are of the type which asserts that the value of the physical quantity lies in some Borel set of the real numbers. The proposition that the value of a physical quantity lies in a Borel set of the real line corresponds to an event as it is apprehended by an observer using his measuring instrument. Thus we obtain a mapping from the Borel sets of the real line to the event structure which captures precisely the notion of observable. We may argue that the real line endowed with its Borel structure serves as a modeling object which schematizes the event algebra of an observed system by projecting its structure into it. In the Hilbert space formalism of quantum theory, events are considered as closed subspaces of a seperable, complex Hilbert space corresponding to a physical system. Then the quantum event algebra is identified with the lattice of closed subspaces of the Hilbert space, ordered by inclusion and carrying an orthocomplementation operation which is given by the orthogonal complements of the closed subspaces [7]. Subsequently a quantum event structure is defined to be the category of quantum event algebras and quantum algebraic homomorphisms. In effect a nonclassical, non-Boolean logical structure is induced which has its origins in quantm theory.

In the quantum-logic approach the notion of event is taken to be equivalent to a proposition of a physical system. This formulation of quantum theory is based on the identification of propositions with projection operators on a complex Hilbert space [8]. Furthermore, the order relations and the lattice operations of the lattice of quantum propositions are associated with the logical implication relation and the logical operations of conjuction, disjunction, and negation of propositions.

In Section 2 we introduce the concepts of variable sets and fibrations. In Section 3 we define the categories associated with event and observable structures. In Section 4 we develop a Boolean manifold perspective on quantum theory based on the idea that observables provide a coordinatization of the quantum world. In Section 5 we analyze the semantics of the attempt to probe the quantum structure through Boolean measurement localization systems. Finally, we conclude in Section 6. An appendix contains a detailed proof of the adjunction we have constructed between presheaves of Boolean observables and quantum observables.

2. VARIABLE SETS AND FIBRATIONS

For a category $\mathscr{A}$ we will be considering the presheaf category $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$ of all contravariant functors from $\mathscr{A}$ to **Sets** and all natural transformations between these. A functor **P** is a structure-preserving morphism of these categories, that is, it preserves composition and identities. A functor in the category $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$ can be thought of as constructing an image of $\mathscr{A}$ in **Sets** contravariantly, or as a contravariant translation of the "language" of $\mathscr{A}$ into that of **Sets**. Given another such translation (contravariant functor) **Q** of $\mathscr{A}$ into **Sets** we need to compare them. This can be done by giving, for each object **A** in $\mathscr{A}$, a transformation τ_A: $\mathbf{P}(A) \to \mathbf{Q}(A)$ which compares the two images of the object A. Not any morphism will do, however, as we would like the construction to be parametric in A rather than ad hoc. Since A is an object in $\mathscr{A}$ whereas $\mathbf{P}(A)$ is in **Sets** we cannot link them by a morphism. Rather, the goal is that the transformation should respect the morphisms of $\mathscr{A}$, or, in other words, the interpretations of v: $A \to C$ by **P** and **Q** should be compatible with the transformation under τ. Then τ is a natural transformation in the presheaf category $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$.

An object **P** of $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$ may be understood as a right action of $\mathscr{A}$ on a set which is partioned into sorts parametrized by the objects of $\mathscr{A}$ and such that whenever $v : C \to A$ is an arrow and p is an element of **P** of sort A, then pv is specified as an element of **P** of sort C such that the following conditions are satisfied:

$$p1_A = p, \qquad p(vw) = (pv)w, \qquad wv : \quad D \to C \to A$$

Such an action **P** is referred as a $\mathscr{A}$-variable set. The fact that any morphism τ: $\mathbf{P} \to \mathbf{Q}$ in the presheaf category $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$ is a natural transformation is expressed by the condition

$$\tau(p, v) = \tau(p)(v)$$

where the first action of v is the one given by **P** and the second by **Q**.

We formalize the above observations as follows: If $\mathscr{A}^{\mathrm{op}}$ is the opposite category of $\mathscr{A}$, then $\mathbf{Sets}^{\mathscr{A}^{\mathrm{op}}}$ denotes the functor category of presheaves on

$\mathscr{A}$, with objects all functors $\mathbf{P}: \mathscr{A}^{op} \to \mathbf{Sets}$ and morphisms all natural transformations between such functors. Each object $\mathbf{P}$ in this category is a contravariant set-valued functor on $\mathscr{A}$, called a presheaf on $\mathscr{A}$.

For each object A of $\mathscr{A}$, $\mathbf{P}(A)$ is a set, and for each arrow $v: C \to A$, $\mathbf{P}(v): \mathbf{P}(A) \to \mathbf{P}(C)$ is a set function. If $\mathbf{P}$ is a presheaf on $\mathscr{A}$ and $p \in \mathbf{P}(A)$, the value $\mathbf{P}(v)(p)$ for an arrow $v: C \to A$ in $\mathscr{A}$ is called the restriction of p along f and is denoted by $\mathbf{P}(v)(p) = p/v$.

Each object A of $\mathscr{A}$ gives rise to a contravariant Hom-functor $y[A] := \mathrm{Hom}_{\mathscr{A}}(—, A)$. This functor defines a presheaf on $\mathscr{A}$. Its action on an object C of $\mathscr{A}$ is given by

$$y[A] := \mathrm{Hom}_{\mathscr{A}}(C, A)$$

whereas its action on a morphism $D \xrightarrow{x} C$ for $v: C \to A$ is given by

$$y[A](x): \quad \mathrm{Hom}_{\mathscr{A}}(C, A) \to \mathrm{Hom}_{\mathscr{A}}(D, A)$$
$$y[A](x)(v) = v \circ x$$

Furthermore, y can be made into a functor from $\mathscr{A}$ to the contravariant functors on $\mathscr{A}$

$$y: \quad \mathscr{A} \to \mathbf{Sets}^{\mathscr{A}^{op}}$$

such that $A \mapsto \mathrm{Hom}_{\mathscr{A}}(—, A)$. This consists of an embedding and it is a full and faithful functor.

There is a set consisting of all the elements of all the sets $\mathbf{P}(A)$, and similarly there is a set consisting of all the functions $\mathbf{P}(v)$. We will formalize these observations about $\mathbf{P}: \mathscr{A}^{op} \to \mathbf{Sets}$ by taking the disjoint union of all the sets of the form $\mathbf{P}(A)$ for all objects A of $\mathscr{A}$. The elements of this disjoint union can be represented as pairs (A, p) for all objects A of $\mathscr{A}$ and elements $p \in \mathbf{P}(A)$. We can say that we construct the disjoint union of sets by labeling the elements. Now we can construct a category whose set of objects is the disjoint union just mentioned. This structure is called the category of elements of $\mathbf{P}$, denoted by $\mathbf{G}(\mathbf{P}, \mathscr{A})$. Its objects are all pairs (A, p), and its morphisms $(\acute{A}, \acute{p}) \to (A, p)$ are those morphisms $u: \acute{A} \to A$ of $\mathscr{A}$ for which $pu = \acute{p}$. Projection on the second coordinate of $\mathbf{G}(\mathbf{P}, \mathscr{A})$, defines a functor $\mathbf{G}(\mathbf{P}): \quad \mathbf{G}(\mathbf{P}, \mathscr{A}) \to \mathscr{A}$. $\mathbf{G}(\mathbf{P}, \mathscr{A})$ together with the projection functor $\mathbf{G}(\mathbf{P})$ is called the split discrete fibration induced by $\mathbf{P}$, and $\mathscr{A}$ is the base category of the fibration. The word "discrete" refers to the fact that the fibers are categories in which the only arrows are identity arrows. If A is an object of

$\mathscr{A}$, the inverse image under $\mathbf{G}(\mathbf{P})$ of A is simply the set $\mathbf{P}(A)$, although its elements are written as pairs so as to form a disjoint union:

$$\begin{array}{ccc} \mathbf{G}(\mathbf{P}, A) & & \\ {\scriptstyle \mathbf{G_P}}\downarrow & & \\ \mathbf{A} & \xrightarrow{\mathbf{P}} & \mathbf{Sets} \end{array} \tag{1}$$

3. CATEGORIES OF EVENT AND OBSERVABLE STRUCTURES

According to the category-theoretic approach, to each species of mathematical structure there corresponds a *category* whose objects have that structure and whose morphisms preserve it. Moreover, to any natural construction on structures of one species yielding structures of another species, there corresponds a *functor* from the category of first species to the category of the second.

A classical event structure is a category, denoted by $\mathscr{B}$, which is called the category of Boolean event algebras. Its objects are Boolean algebras of events and its arrows are Boolean algebraic homomorphisms.

A quantum event structure is a category, denoted by $\mathscr{L}$, which is called the category of quantum event algebras. Its objects are quantum algebras of events, that is, partially ordered sets of quantum events endowed with a maximal element 1 and with an operation of orthocomplementation $[—]^*: L \to L$, which satisfy, for all $l \in L$ the following conditions: [a] $l \leq 1$, [b] $l^{**} = l$, [c] $l \vee l^* = 1$, [d] $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$, [e] $l \perp \acute{l} \Rightarrow l \vee \acute{l} \in L$, and [f] $l \vee \acute{l} = 1$, $l \wedge \acute{l} = 0 \Rightarrow l = \acute{l}^*$, where $0 := 1^*$. $l \perp \acute{l} := l \leq \acute{l}^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usual.

Its arrows are quantum algebraic homomorphisms, that is, maps $L \overset{H}{\to} K$, which satisfy, for all $k \in K$, the following conditions: [a] $H(1) = 1$, [b] $H(k^*) = [H(k)]^*$, [c] $k \leq \acute{k} \Rightarrow H(k) \leq H(\acute{k})$, and [d] $k \perp \acute{k} \Rightarrow H(k \vee \acute{k}) \leq H(k) \vee H(\acute{k})$.

Next we introduce the categories associated with structures of observables.

A quantum observable space structure is a category, denoted by $\mathbb{O}$B, which is called the category of spaces of quantum observables.

Its objects are the sets Ω of real-valued observables on a quantum event algebra L, where each observable Ξ is defined to be an algebraic homomorphism from the Borel algebra of the real line Bor($\mathbf{R}$), to the quantum event algebra L,

$$\Xi : \quad \mathrm{Bor}(\mathbf{R}) \to L$$

such that the following conditions are satisfied: [i] $\Xi(\emptyset) = 0$, $\Xi(\mathbf{R}) = 1$, [ii] $E \cap F = \emptyset \Rightarrow \Xi(E) \perp \Xi(F)$ for $E, F \in \mathrm{Bor}(\mathbf{R})$, [iii] $\Xi(\cup_n E_n) =$

$\vee_n \Xi(E_n)$, where $E_1, E_2, \ldots$ is a sequence of mutually disjoint Borel sets of the real line.

If L is isomorphic with the orthocomplemented lattice of orthogonal projections on a Hilbert space, then it follows from von Neumann's spectral theorem that the observables are in 1–1 correspondence with the hypermaximal Hermitian operators on the Hilbert space.

Moreover, each set Ω is endowed with a right action R: $\Omega \times \text{Bor}\, f(\mathbf{R}) \to \Omega$ from the semigroup of all real-valued Borel functions of a real variable f: $\mathbf{R} \to \mathbf{R}$ which satisfy the following condition:

$$E \in \text{Bor}(\mathbf{R}) \Rightarrow f^{-1}(E) \in \text{Bor}(\mathbf{R})$$

According to the above, we have

$$(\Xi, f) \in \Omega \times \text{Bor}\, f(\mathbf{R}) \mapsto \Xi \bullet f = \Xi(f^{-1}(E)) \in \Omega$$

To sum up, the objects of the category of quantum observables are the spaces $\mathbf{\Omega} = \langle \Omega, \mathbf{R} \rangle$ of real-valued observables.

Its arrows are the quantum observable spaces homomorphisms h: $\mathbf{\Omega} \to \mathbf{U}$, namely set-homomorphisms $[\cdot]^h$: $\Omega \to U$ which respect the right action of Bor $f(\mathbf{R})$:

$$[\Xi \bullet f]^h = \Xi^h \bullet f$$

We note that $\mathbf{\Omega}$ and $\mathbf{U}$ are regarded as defined over the same quantum event algebra L; otherwise we have to take into account the quantum algebraic homomorphisms as well.

Using the information encoded in the categories of quantum event algebras $\mathcal{L}$ and spaces of quantum observables $\mathbb{O}B$, it is possible to construct a new category, called the category of quantum observables, which is going to play a key role in the subsequent analysis, and is defined as follows:

A quantum observable structure is a category, denoted by $\mathbb{O}_Q$, which is called the category of quantum observables.

Its objects are the quantum observables Ξ: Bor($\mathbf{R}$) $\to L$ and its arrows $\Xi \to \Theta$ are the commutative triangles

$$\begin{array}{ccc} & \text{Bor}(R) & \\ {}^{\Xi}\swarrow & & \searrow^{\Theta} \\ L & \xrightarrow{\quad H \quad} & K \end{array} \qquad (2)$$

or equivalently the quantum algebraic homomorphisms $L \xrightarrow{H} K$ in $\mathcal{L}$, such that $\Theta = H \circ \Xi$ in the above diagram is again a quantum observable.

Correspondingly, a Boolean observable structure is a category, denoted by $\mathbb{O}_B$, which is called the category of Boolean observables.

Its objects are the Boolean observables ξ: Bor($\mathbf{R}$) $\rightarrow B$ and its arrows are the Boolean algebraic homomorphisms $B \xrightarrow{h} C$ in $\mathcal{B}$ such that $\theta = h \circ \xi$ in

$$\begin{array}{ccc} & \text{Bor}(R) & \\ {}^{\xi}\swarrow & & \searrow^{\theta} \\ B & \xrightarrow{\quad h \quad} & C \end{array} \qquad (3)$$

is again a Boolean observable.

The categories of event algebras and observables are related functorialy as follows: Under the action of a shaping functor Bor**R** may be considered as an object of $\mathcal{L}$. Hence it is possible to construct the covariant representable functor $\mathbf{F}$: $\mathcal{L} \rightarrow$ **Sets** defined by $\mathbf{F} = \text{Hom}_{\mathcal{L}}(\text{Bor}\mathbf{R}, \text{—})$. The aplication of the fibration technique on the functor **F** provides the category of elements of this functor, which is the category of all arrows in $\mathcal{L}$ from the object Bor**R**, characterized equivalently as the comma category [Bor**R**/$\mathcal{L}$]. We conclude that the category of quantum observables $\mathcal{O}_Q$ is actually the comma category [Bor**R**/$\mathcal{L}$] or equivalently the category of elements of the functor $\mathbf{F} = \text{Hom}_{\mathcal{L}}(\text{Bor}\mathbf{R}, \text{—})$. Analogous comments hold for the category of Boolean observables.

4. A BOOLEAN MANIFOLD PERSPECTIVE ON QUANTUM THEORY

The fact that the categories of quantum and Boolean observables can be characterized as the comma categories [Bor**R**/$\mathcal{L}$] and [Bor**R**/$\mathcal{B}$], respectively, permits us to apply the construction of ref. 1 in the context of the above categories and validate an interpretational perspective on quantum theory according to which the quantum world is being perceived through Boolean reference frames, objectified by measuring arrangements set up experimentally, which interlock with each other nontrivially. Thus we procceed as follows.

4.1. Adjointness

As a first step we define a shaping or modeling or coordinatization functor $\mathbf{A}$: $\mathcal{O}_B \rightarrow \mathcal{O}_Q$ which assigns to Boolean observables in $\mathcal{O}_B$ (which plays the role of the model category) the underlying quantum observables from $\mathcal{O}_Q$ and to Boolean homomorphisms the underlying quantum homomorphisms.

Equivalently the shaping functor can be characterized as $\mathbf{A}$: $\mathcal{B} \rightarrow \mathcal{L}$, which assigns to Boolean event algebras in $\mathcal{B}$ (which plays the role of the model category) the underlying quantum event algebras from $\mathcal{L}$ and to Boolean homomorphisms the underlying quantum algebraic homomorphisms, such that

$$\begin{array}{ccc} & \mathrm{Bor}(R) & \\ {}^{\xi}\swarrow & & \searrow^{\Xi} \\ \mathbf{A}(B_{\Xi}) & \xrightarrow{[\Psi_B]_{\Xi}} & L \end{array} \tag{4}$$

commutes.

We consider the category of quantum observables $[\mathrm{Bor}\mathbf{R}/\mathcal{L}]$ and the modeling functor $\mathbf{A}$, and we define the functor $\mathbf{R}$ from $[\mathrm{Bor}\mathbf{R}/\mathcal{L}]$ to presheaves given by

$$\mathbf{R}(\Xi): \quad \xi \mapsto \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathcal{L}]}\,(\mathbf{A}(\xi), \Xi)$$

The functor $\mathbf{R}(\Xi)$ serves to prove the existence of an adjunction expressed by the following bijection, natural in $\mathbf{P}$ and Ξ (a detailed presentation of the adjunction we have constructed is provided in the Appendix):

$$\mathrm{Nat}(\mathbf{P}, \mathbf{R}(\Xi)) \cong \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathcal{L}]}(\mathbf{LP}, \Xi)$$

where the left adjoint $\mathbf{L}$: $\mathbf{Sets}^{[[\mathrm{Bor}\mathbf{R}/\mathcal{B}]]^{\mathrm{op}}} \to [\mathrm{Bor}\mathbf{R}/\mathcal{L}]$ is defined for each presheaf of Boolean observables $\mathbf{P}$ in $Sets^{[[\mathrm{Bor}\mathbf{R}/\mathcal{B}]]^{\mathrm{op}}}$ as the colimit taken in the category of elements of $\mathbf{P}$,

$$\mathbf{L}(\mathbf{P}) = \mathrm{Colim}\{\mathbf{G}(\mathbf{P}, [\mathbf{BorR}/\mathcal{B}]) \to [\mathbf{BorR}/\mathcal{B}] \to [\mathbf{BorR}/\mathcal{L}]\}$$

Furthermore, it has been shown in ref. 1 that the categorical construction of this colimit as a coequalizer of a coproduct reveals the fact that this left adjoint is like the tensor product $-\otimes_{[\mathrm{Bor}\mathbf{R}/\mathcal{B}]} \mathbf{A}$ (Appendix).

Consequently there is a pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$ as follows:

$$\mathbf{L}: \quad \mathbf{Sets}^{[[\mathrm{Bor}\mathbf{R}/\mathcal{B}]]^{\mathrm{op}}} \leftrightarrows [\mathrm{Bor}\mathbf{R}/\mathcal{L}]: \quad \mathbf{R}$$

4.2. Systems of Boolean Measurement Charts

In order to extract the physical meaning it is necessary to proceed through the second step of the proposed construction. The second step is the introduction of the notion of a system of measurement localizations for observable Ξ over quantum event algebra L in $[\mathrm{Bor}\mathbf{R}/\mathcal{L}]$. This amounts to the consideration that $\mathbf{P}$ is a subfunctor of the Hom-functor $\mathbf{R}(\Xi)$ of the form $\mathbf{S} : [[\mathrm{Bor}\mathbf{R}/\mathcal{B}]]^{\mathrm{op}} \to \mathbf{Sets}$, namely for all ξ in $[\mathrm{Bor}\mathbf{R}/\mathcal{B}]$ it satisfies $\mathbf{S}(\xi) \subseteq [\mathbf{R}(\Xi)](\xi)$.

Equivalently it may be described as a set $\mathbf{S}(B)$ of maps of the form

$$\psi_{\xi}: \quad \mathbf{A}(\xi) \to \Xi, \qquad \xi \in [\mathrm{Bor}\mathbf{R}/\mathcal{B}]$$

such that $\langle \psi_{\xi}$: $\mathbf{A}(\xi) \to \Xi$ in $\mathbf{S}(\xi)$ and $\mathbf{A}(v)$: $\mathbf{A}(\xi) \to \mathbf{A}(\acute{\xi})$ in $[\mathrm{Bor}\mathbf{R}/\mathcal{L}]$ for v: $\xi \to \acute{\xi}$ in $[\mathrm{Bor}\mathbf{R}/\mathcal{B}]$ implies $\psi_{\xi} \circ \mathbf{A}(v)$: $\mathbf{A}(\acute{\xi}) \to [\mathrm{Bor}\mathbf{R}/\mathcal{L}]$ in $\mathbf{S}(\xi)\rangle$. In turn, a system of measurement localizations for observable Ξ over quantum event algebra L in $[\mathrm{Bor}\mathbf{R}/\mathscr{L}]$ is equivalent to a system of measurement localizations $(B_\Xi, [\psi_B]_\Xi \colon \mathbf{A}(B_\Xi) \to L)$ for quantum event algebra L in $\mathscr{L}$ making

$$\begin{array}{ccc} \mathrm{Bor}(\mathbf{R}) & \xrightarrow{\acute{\xi}} & \mathbf{A}(\acute{B}_\Xi) \\ \xi\downarrow & \overset{\Xi}{\searrow} & \downarrow [\psi_{\acute{B}}]_\Xi \\ \mathbf{A}(\boldsymbol{B}_\Xi) & \xrightarrow{[\psi_B]_\Xi} & L \end{array} \tag{5}$$

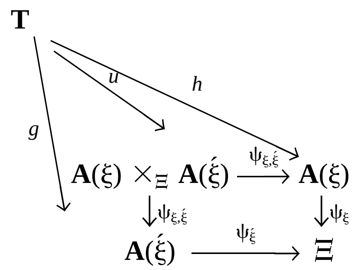
commutative.

The introduction of this notion makes the functioning of the adjunction clear, as we will explain below, and, moreover, can be fruitfully used to obtain our objective, a Boolean manifold representation of quantum event algebra L induced by measurement reference frames for observable Ξ over L.

The motivation behind the concept of a system of measurement localizations has concrete physical grounds. According to the Kochen–Specker theorem [9], it is not possible to understand completely a quantum mechanical system with the use of a single system of Boolean devices. On the other hand, in every concrete experimental context, the set of events that have been actualized in this context forms a Boolean algebra. In the light of this we can say that any Boolean object $(B_\Xi, [\psi_B]_\Xi \colon \mathbf{A}(B_\Xi) \to L)$ in a system of prelocalizations for quantum event algebra making diagram (5) commutative corresponds to a set of Boolean classical events that become actualized in the experimental context of B. These Boolean objects deserve the name measurement-shaping objects. The above observation is equivalent to the statement that a measurement Boolean algebra serves as a reference frame relative to which a measurement result is being coordinatized. Correspondingly, by diagram (5), we obtain naturally the notion of coordinatizing Boolean observables in a system of prelocalizations for a quantum observable over quantum event algebra L. We may even advance it to the status of a "principle of relativity" in quantum physics, suggesting a way of viewing its formalism in a relativistic and contextualistic perspective. Philosophically speaking, we can assert that the quantum world is being perceived through Boolean reference frames, regulated by our measurement procedures, which interlock to form a coherent picture in a nontrivial way.

In this perspective the role of the Hom-functor $\mathbf{R}(\Xi)$ is to single out a set of algebraic homomorphisms which play the role of local coverings of a quantum observable by modeling objects. The notion of a system of prelocalizations boils down essentially to sending many Boolean observables into the quantum observable homomorphically, expecting that these modeling objects will prove to be sufficient for determination of the quantum observable. If we consider the point of view offered by manifold theory, we may characterize the maps ψ_ξ: $\mathbf{A}(\xi) \to \Xi$, $\xi \in [\mathrm{Bor}\mathbf{R}/\mathcal{B}]$, in a system of prelocalizations for quantum observable Ξ as Boolean observable charts. Correspondingly, the shaping Boolean objects $(B_\Xi, [\psi_B]_\Xi: \mathbf{A}(B_\Xi) \to L)$ in a system of prelocalizations for quantum event algebra making diagram (5) commutative may be characterized as measurement charts. In turn, their domains B_Ξ may be called Boolean coordinate domains for measurement, the elements of B_Ξ measured Boolean coordinates, and the elements of L as quantum events or quantum propositions. The Boolean homomorphisms v: $B_\Xi \to \acute{B}_\Xi$ in $\mathcal{B}$ may be characterized as transition maps.

Moreover, the pullback of the Boolean charts ψ_ξ: $\mathbf{A}(\xi) \to \Xi$, $\xi \in [\mathrm{Bor}\mathbf{R}/\mathcal{B}]$, and $\psi_{\acute{\xi}}$: $\mathbf{A}(\acute{\xi}) \to \Xi$, $\acute{\xi} \in [\mathrm{Bor}\mathbf{R}/\mathcal{B}]$, with common codomain the quantum observable Ξ consists of the object $\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\acute{\xi})$ and two arrows $\psi_{\xi\acute{\xi}}$ and $\psi_{\acute{\xi}\xi}$, called projections, as shown in



(6)

The square commutes and for any object T and arrows h and g that make the outer square commute there is a unique u: $T \to \mathbf{A}(\xi) \times_\Xi \mathbf{A}(\acute{\xi})$ that makes the whole diagram commute. Hence we obtain the condition

$$\psi_{\acute{\xi}} \circ g = \psi_\xi \circ h$$

The pullback of the Boolean charts ψ_ξ: $\mathbf{A}(\xi) \to \Xi$, $\xi \in [\mathrm{Bor}\mathbf{R}/\mathcal{B}]$, and $\psi_{\acute{\xi}}$: $\mathbf{A}(\acute{\xi}) \to \Xi$, $\ddot{\xi} \in [\mathrm{Bor}\mathbf{R}/\mathcal{B}]$, is equivalently characterized as their fiber product because $\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\acute{\xi})$ is not the whole product $\mathbf{A}(\xi) \times \mathbf{A}(\acute{\xi})$, but the product taken fiber by fiber. We notice that if ψ_ξ and $\psi_{\acute{\xi}}$ are 1–1, then their pullback is isomorphic with the intersection $\mathbf{A}(\xi) \cap \mathbf{A}(\acute{\xi})$. Then we can define the pasting map, which is an isomorphism, as

$$\Omega_{\xi,\acute{\xi}}: \quad \psi_{\acute{\xi}\xi}(\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\acute{\xi})) \to \psi_{\xi\acute{\xi}}(\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\acute{\xi}))$$

by putting

$$\Omega_{\xi,\acute{\xi}} = \psi_{\xi\acute{\xi}} \circ \psi_{\acute{\xi}\xi}^{-1}$$

Then we have the following conditions:

$$\Omega_{\xi,\xi} = 1_{\xi} \qquad 1_{\xi}: \text{ identity of } \xi$$

$$\Omega_{\xi,\acute{\xi}} \circ \Omega_{\acute{\xi},\acute{\acute{\xi}}} = \Omega_{\xi,\acute{\acute{\xi}}} \qquad \text{if} \quad \mathbf{A}(\xi) \cap \mathbf{A}(\acute{\xi}) \cap \mathbf{A}(\acute{\acute{\xi}}) \neq 0$$

$$\Omega_{\xi,\acute{\xi}} = \Omega_{\acute{\xi},\xi} \qquad \text{if} \quad \mathbf{A}(\xi) \cap \mathbf{A}(\acute{\xi}) \neq 0$$

The pasting map ensures that $\psi_{\acute{\xi}\xi}(\mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\acute{\xi}))$ and $\psi_{\xi\acute{\xi}}(\mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\acute{\xi}))$ cover the same part of the quantum observable in a compatible way.

It is obvious that the above compatibility conditions are translated immediately into corresponding compatibility conditions concerning Boolean measurement charts on the quantum event structure.

4.3. Boolean Manifolds of Measurement Charts

The representation of a quantum event algebra L as a Boolean manifold composed of measurement Boolean charts is completed iff the counit of the adjunction between presheaves of Boolean observables and quantum observables according to the vertical map in

$$\begin{array}{ccccc} \coprod_{v:\acute{\xi}\to\xi} \mathbf{A}(\acute{\xi}) & \overset{\zeta}{\underset{\eta}{\rightrightarrows}} & \coprod_{(\xi,p)} \mathbf{A}(\xi) & \to & [\mathbf{R}(\Xi)](-) \otimes_{\mathbb{C}_B} \mathbf{A} \\ & & & \searrow & \Big\downarrow{\scriptstyle \epsilon_{\Xi}} \\ & & & & \Xi \end{array} \tag{7}$$

restricted to subfunctors of the Hom-functor $\mathbf{R}(\Xi)$ is an isomorphism, namely structure-preserving, 1–1, and onto. The significance of this representation lies in the fact that it is possible to dispense with the structure of a quantum event algebra L completely and use instead one of the Boolean measurement localization systems for observable Ξ over L characterized equivalently as measurement Boolean atlases.

If we focus our attention on a Boolean measurement localization system for quantum observable Ξ over quantum event algebra L, we observe that the objects of the category of elements $\mathbf{G}(\mathbf{R}(\Xi), \xi)$ are actually the aforementioned local modeling Boolean measurement charts and its maps are the transition functions.

In other words, the objects of the category of elements $\mathbf{G}(\mathbf{R}(\Xi), \xi)$ are pairs $(\xi, \psi_{\xi}\colon \mathbf{A}(\xi) \to \Xi)$ with ξ in $[\mathrm{Bor}\mathbf{R}/\mathcal{B}]$ and ψ_{ξ} an arrow in $[\mathrm{Bor}\mathbf{R}/\mathcal{L}]$, namely a quantum homomorphism. Similarly, the objects of the corresponding category of elements $\mathbf{G}(\mathbf{R}(L), B)$ are pairs $(B_{\Xi}, [\psi_B]_{\Xi}\colon \mathbf{A}(B_{\Xi}) \to L)$ in a system of measurement localizations for observable Ξ over quantum event algebra L making diagram (5) commutative.

A morphism $(\acute{B}, \psi_{\acute{B}}) \to (B, \psi_B)$ in the above category of elements (for notational convenience the index Ξ is not stated explicitly) is an arrow v: $\acute{B} \to B$ in [Bor($\mathbf{R}$)/$\mathcal{B}$], namely a Boolean homomorphism, with the property that $\psi_{\acute{B}} = \psi_B \circ \mathbf{A}(v)$: $\mathbf{A}(\acute{B}) \to L$; in other words, v must take the chosen measurement Boolean chart ψ_B in $\mathbf{G}(\mathbf{R}(L), B)$ back into $\psi_{\acute{B}}$ in $\mathbf{G}(\mathbf{R}(\acute{L}), \acute{B})$. These morphisms are composed by composing the underlying arrows v of [Bor $\mathbf{R}$/$\mathcal{B}$].

It is instructive to emphasize that each measurement Boolean chart may be characterized as a B_Ξ-parametrized family or equivalently as a varying element, in that if we evaluate it at various stages, we will vary it through various points of L. Thus the measurement Boolean charts effect a naming or coordinatization of elements of L by B_Ξ, emphasizing the fact that each map $[\psi_B]_\Xi$: $\mathbf{A}(B_\Xi) \to L$ produces a structure in L. In this perspective it is clear that when a measurement Boolean chart is considered as a figure of shape B_Ξ in L, we think of L as a fixed object and of $\mathbf{A}(B_\Xi)$ as variable so as to give all possible shapes of figures in L. Furthermore, the compatibility relations that the measurement Boolean charts obey determine to what extent the corresponding figures overlap and what the structure of this overlap is.

In addition, if L is conceived of as a truth-value structure, each Boolean measurement chart can be understood as an unsharp or fuzzy Boolean algebra of events corresponding to measurement of observable Ξ. More concretely, since these generalized elements are maps $[\psi_B]_\Xi$: $\mathbf{A}(B_\Xi) \to L$, each Boolean event in B_Ξ, besides its "yes" or "no" occurrence, will have other intermediate stages of occurrence, measured by the degrees in the lattice L in $\mathcal{L}$.

The counit of the adjunction being surjective means that the measurement Boolean charts in $\mathbf{G}(\mathbf{R}(L), B_\Xi)$ cover entirely the quantum event algebra L. The counit being injective means that any two measurement Boolean charts are always compatible. Moreover, the counit also being an algebraic homomorphism means that it preserves the structure, hence in effect the quantum algebra L is determined completely by the measurement Boolean charts and their compatibility relations in a system of its localizations. Each chart corresponds to a set of measured Boolean events locally related to observable Ξ. The equivalence classes of measurement charts represent the same quantum events in L. We observe that since two different local Boolean measurement charts can overlap, we have the possibility of observing quantum events from different reference frames, but due to the equivalence and compatibility relations these different observational contexts are equivalent and moreover establish the same quantum event. In this perspective quantum events may be interpreted as fuzzy Boolean events locally, and quantum observables as fuzzy Boolean observables locally.

5. SEMANTICS OF BOOLEAN MEASUREMENT LOCALIZATION SYSTEMS

The measurement Boolean reference frame perspective on quantum structure is based on using observable coordinatizing objects belonging to the Boolean species of structure as shaping figures to probe objects belonging to the quantum species of structure. The modeling objects give rise to structure-preserving maps with the modeling objects as their domains, which fit compatibly together in a system of localizations, effecting an isomorphism between quantum event algebra objects and Boolean measurement localization systems. Consequently the structure of the quantum event algebra is recovered by the information that its structure-preserving maps, encoded as Boolean measurement charts in localization systems, carry, as well as their compatibility relations. This leads naturally to a relativistic, contextualistic conception of quantum events with respect to Boolean reference frames of measurement, and finally to a representation of them as equivalence classes of measured Boolean fuzzy events. Equivalently, quantum observables are understood through isomorphism classes of their Boolean localizations on measurement charts.

The roots of the above contextualistic perspective on quantum structure established by systems of Boolean measurement localization systems are located in the physical meaning of the adjunction between presheaves of Boolean observables and quantum observables under the condition that the variation in the environment of the category of presheaves is arrested on the Hom-functor $\mathbf{R}(L)$.

The transition from classical to quantum physics essentially involves the transition from Boolean event structures to non-Boolean event structures, or from those that do to those that do not admit two-valued homomorphisms. An observable schematizes the quantum event structure by correlating its Boolean figures picked by measurements with the smallest Boolean algebra containing all the clopen sets of the real line. Thus Boolean observables play the role of coordinatizing objects in the attempt to probe the quantum world by picking Boolean figures and subsequently opening Boolean windows for the perception of the latter, interpreted as measurement charts. Let $\mathbf{Sets}^{\mathcal{B}^{op}}$ be the world of Boolean observable event structures modeled in **Sets** by an observer, or the world of Boolean windows, and $\mathcal{L}$ that of quantum event structures. In this perspective the functor $\mathbf{L}\colon \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$ can be conceived of as a translational code from Boolean windows to the quantum species of event structure, whereas the functor $\mathbf{R}\colon \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}}$ can be thought of as a translational code in the inverse direction. It is impossible to expect that translating from one language to another and back will leave the meaning unchanged. However, there remain two ways for a Boolean-event algebra variable set **P**, or Boolean window, to communicate a message to a quantum event algebra L. Either the information is given in quantum terms with **P** translating, which we can represent as $\mathbf{LP} \to L$, or the information is given in Boolean terms with L translating, represented as $\mathbf{P} \to \mathbf{R}(L)$. In the first case L may think that it receives information in quantum terms, while in the second, **P** may think it sends information in Boolean terms. The natural bijection then corresponds to the assertion that these two distinct ways of communicating are equivalent. In this perspective the left adjunction operator can be characterized as the *quantization functor* and the right adjunction operator as the *classicalization functor.*

We have argued that we can probe the quantum world through Boolean windows, interpreted as measurement charts. Furthermore, we may apply Stone's representation theorem for Boolean algebras, according to which it is legitimate to replace Boolean algebras by fields of subsets of a sample space. Hence if we replace each Boolean algebra B_Ξ making diagram (5) commutative by its set-theoretic representation $[\Sigma, B_\Sigma]$ consisting of a local sample space Σ and its local field of subsets B_Σ, it is possible to define local measurement space charts $(B_\Sigma, \psi_{B_\Sigma}: \mathbf{A}(B_\Sigma) \to L)$ and corresponding localization systems for quantum observable Ξ in $[\mathrm{Bor}\mathbf{R}/\mathscr{L}]$. We note that the inverse of $\xi = \Xi_B$ plays the role of a classical random variable on Σ. Thus every quantum observable may be treated locally as a classical random variable.

The equivalence relation for measurement Boolean charts for observable Ξ over quantum event algebra L establishes a criterion for the equivalence of experimental contexts, or measurement procedures. Thus, using Boolean measurement charts $(B, \psi_B: \mathbf{A}(B) \to L)$ such that diagram (5) commutes and modeling Boolean coordinates $b \in B$, we can form their equivalence classes, which, modulo the compatibility conditions on overlaps, will represent a single quantum event in L. The fact that two different measurement Boolean charts $(B, \psi_B: \mathbf{A}(B) \to L)$ and $(C, \psi_C: \mathbf{A}(C) \to L)$ in a system of localizations can overlap reflects the possibility of using two distinct experimental contexts, or equivalently setting up two different measurement procedures, and hence obtaining two different outcomes registered by the Boolean coordinates $b \in B$ and $c \in C$. At this point, the equivalence relation between measurement Boolean charts informs us that that these experimental contexts are in fact equivalent, and furthermore that, since in a system of localizations compatibility relations on overlaps are satisfiable, the same quantum event is being verified. Equivalently, we can say that we identify those experimental outcomes whose underlying local Boolean observables are related by the established equivalence relation of the tensor product construction, and diagram (5) commutes as well.

Correspondingly, from local measurement sample space charts $(B_\Sigma, \psi_{B_\Sigma}: \mathbf{A}(B_\Sigma) \to L)$ for observable Ξ over quantum event algebra L we may form their equivalence classes, which, modulo the conditions for compatibility on overlaps, will represent a single quantum event in L, corresponding to the quantum observable Ξ. Under these circumstances we may interpret the equivalence classes of measurement local space charts $\psi_{B_\Sigma} \otimes a$, $a \in \mathbf{A}(B_\Sigma)$, such that the commutativity of diagram (5) is satisfied, as the statistical experimental actualizations of the quantum events in $\mathcal{L}$, corresponding to observables over $\mathcal{L}$. The local measurement space charts $(B_\Sigma, \psi_{B_\Sigma}: \mathbf{A}(B_\Sigma) \to L)$ and $(C_\Sigma, \psi_{C_\Sigma}: \mathbf{A}(C_\Sigma) \to L)$ are compatible in a system of measurement localizations for observable Ξ over quantum event algebra L iff for some $(D_\Sigma, \psi_{D_\Sigma}: \mathbf{A}(D_\Sigma) \to L)$ in the system of localizations and $a \in \mathbf{A}(B_\Sigma)$, $b \in \mathbf{A}(C_\Sigma)$, c, $d \in \mathbf{A}(D_\Sigma)$, the following conditions hold:

$$\psi_{B_\Sigma} \otimes a = \psi_{D_\Sigma} \otimes c$$

$$\psi_{C_\Sigma} \otimes b = \psi_{D_\Sigma} \otimes d$$

Finally, the pullback compatibility condition may be interpreted in the measurement context as denoting that two local space representations of quantum events corresponding to quantum observables satisfy the compatibility condition on overlaps iff the measurements of observables are equivalent to measurements taken in a common experimental setup.

6. CONCLUSIONS

In an attempt to probe the objects belonging to the quantum species of structure, we have developed the idea of using observables of the Boolean species of structure as coordinatizing objects in the quantum world, which results in a contextualistic perspective on the latter through local Boolean measurement reference frames. An observable effects a schematization of the quantum event structure by correlating Boolean algebras picked by measurements with the Borel algebra of the real line. Thus Boolean observables play the role of coordinatizing objects in the quantum world by picking Boolean figures and subsequently opening Boolean windows for the perception of the latter, interpreted as local measurement charts. The coordinatizing objects give rise to structure-preserving maps with the modeling objects as their domains, which give rise to systems of compatible measurement localizations, effecting finally an isomorphism between quantum event algebra objects and Boolean measurement localization systems. Consequently, the structure of a quantum event algebra is completely recovered by the information that its structure-preserving maps, encoded as Boolean measurement charts in localization systems, carry, as well as their compatibility relations. This leads naturally to a relativistic, contextualistic conception of quantum events with respect to Boolean reference frames of measurement, and finally to a representation of them as equivalence classes of measured Boolean fuzzy events. Equivalently, quantum observables are understood through isomorphism classes of their Boolean localizations on measurement charts.

APPENDIX

We consider the category of quantum observables $[\mathrm{Bor}\mathbf{R}/\mathscr{L}]$ and the modeling functor $\mathbf{A}$, and we define the functor $\mathbf{R}$ from $[\mathrm{Bor}\mathbf{R}/\mathscr{L}]$ to presheaves given by

$$\mathbf{R}(\Xi): \quad \xi \mapsto \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathscr{L}]}(\mathbf{A}(\xi), \Xi)$$

A natural transformation τ between the presheaves on the category of Boolean observables $\mathbf{P}$ and $\mathbf{R}(\Xi)$, $\tau: \mathbf{P} \to \mathbf{R}(\Xi)$, is a family τ_ξ indexed by Boolean observables ξ of $[\mathrm{Bor}\mathbf{R}/\mathscr{B}]$ for which each τ_ξ is a map

$$\tau_\xi: \quad \mathbf{P}(\xi) \to \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathscr{L}]}(\mathbf{A}(\xi), \Xi)$$

of sets, such that the diagram of sets

$$\begin{array}{ccc} \mathbf{P}(\xi) & \xrightarrow{\tau_\xi} & \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathscr{L}]}(\mathbf{A}(\xi, \Xi) \\ \mathbf{P}(u)\downarrow & & \downarrow{}^{*}\mathbf{A}(u) \\ \mathbf{P}(\acute{\xi}) & \xrightarrow{\tau_\xi} & \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathscr{L}]}(\mathbf{A}(\acute{\xi}), \Xi) \end{array} \tag{8}$$

commutes for each Boolean homomorphism $u: \acute{\xi} \to \xi$ of $[\mathrm{Bor}\mathbf{R}/\mathscr{B}]$.

If we use the category of elements of the Boolean observables-variable set P, then the map τ_ξ defined above can be characterized as

$$\tau_\xi: \quad (\xi, p) \to \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathscr{L}]}(\mathbf{A} \circ G_{\mathbf{P}}(\xi, p), \Xi)$$

Equivalently, such a τ can be seen as a family of arrows of $[\mathrm{Bor}\mathbf{R}/\mathscr{L}]$ which is indexed by objects (ξ, p) of the category of elements of the presheaf of Boolean observables $\mathbf{P}$, namely

$$\{\tau_\xi(p): \mathbf{A}(\xi) \to \Xi\}_{(\xi, p)}$$

From the perspective of the category of elements of $\mathbf{P}$, the condition of the commutativity of diagram (8) is equivalent to the condition that for each arrow u,

$$\begin{array}{ccccc} \mathbf{A}(\xi) & = & \mathbf{A}\circ\mathbf{G_P}(\xi, p) & & \\ & & & \searrow^{\tau_\xi(p)} & \\ \downarrow{\scriptstyle \mathbf{A}(u)} & & \downarrow{\scriptstyle u_*} & & \Xi \\ & & & \nearrow_{\acute{\tau}_\xi(p)} & \\ \mathbf{A}(\acute{\xi}) & = & \mathbf{A}\circ\mathbf{G_P}(\acute{\xi}, \acute{p}) & & \end{array} \qquad (9)$$

commutes.

From diagram (9) we can see that the arrows $\tau_\xi(p)$ form a cocone from the functor $\mathbf{A}\circ\mathbf{G_P}$ to the quantum observable algebra object Ξ. Making use of the definition of the colimit, we conclude that each such cocone emerges by the composition of the colimiting cocone with a unique arrow from the colimit **LX** to the quantum observable object Ξ. In other words, there is a bijection which is natural in **X** and Ξ

$$\mathrm{Nat}(\mathbf{P}, \mathbf{R}(\Xi)) \cong \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathscr{L}]}(\mathbf{LX}, \Xi)$$

From the above bijection we are driven to the conclusion that the functor **R** from $[\mathrm{Bor}\mathbf{R}/\mathscr{L}]$ to presheaves given by

$$\mathbf{R}(\Xi): \quad \xi \mapsto \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathscr{L}]}(\mathbf{A}(\xi), \Xi)$$

has a left adjoint $\mathbf{L}$: $\mathbf{Sets}^{[\mathrm{Bor}\mathbf{R}/\mathscr{B}]^{op}} \to [\mathrm{Bor}\mathbf{R}/\mathscr{L}]$, which is defined for each presheaf of Boolean observables **X** in $\mathbf{Sets}^{[\mathrm{Bor}\mathbf{R}/\mathscr{B}]^{op}}$ as the colimit

$$\mathbf{L}(\mathbf{P}) = \mathrm{Colim}\{\mathbf{G}(\mathbf{P}, [\mathbf{BorR}/\mathscr{B}]) \overset{\mathbf{G_P}}{\to} [\mathbf{BorR}/\mathscr{B}] \overset{\mathbf{A}}{\to} [\mathbf{BorR}/\mathscr{L}]\}$$

Consequently, there is a pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$ as follows:

$$\mathbf{L}: \quad \mathbf{Sets}^{[\mathrm{Bor}\mathbf{R}/\mathscr{B}]^{op}} \leftrightarrows [\mathrm{Bor}\mathbf{R}/\mathscr{L}]: \quad \mathbf{R}$$

Thus we have constructed an adjunction which consists of the functors **L** and **R**, called left and right adjoints with respect to each other, respectively, as well as the natural bijection

$$\mathrm{Nat}(\mathbf{P}, \mathbf{R}(\Xi)) \cong \mathrm{Hom}_{[\mathrm{Bor}\mathbf{R}/\mathscr{L}]}(\mathbf{LX}, \Xi)$$

Furthermore, the content of the adjunction can be analyzed if we use the categorical construction of the colimit defined above, as a coequalizer of a coproduct [1]. The coequalizer presentation of the colimit shows that the "Hom-functor" $\mathbf{R}_A$ has a left adjoint which can be characterized categorically as the tensor product $-\otimes_{\mathcal{O}_\mathscr{B}} \mathbf{A}$:

$$\coprod_{v:\acute{\xi}\to\xi}\mathbf{A}(\acute{\xi}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{(\xi,p)}\mathbf{A}(\xi) \overset{\chi}{\to} \mathbf{X}\otimes_{\mathbb{O}_b}\mathbf{A} \tag{10}$$

In diagram (10) the second coproduct is over all the objects (ξ, p) with $p \in \mathbf{X}(\xi)$ of the category of elements, while the first coproduct is over all the maps $v: (\acute{\xi}, \acute{p}) \to (\xi, p)$ of that category, so that $v: \acute{\xi} \to \xi$ and the condition $pv = \acute{p}$ is satisfied.

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